Math 72

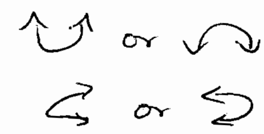
Chapter 10 Overview: CONIC SECTIONS

10.1 Parabola and Circle
10.2 Ellipse and Hyperbola
10.3 Systems of Nonlinear Equations and Systems of Nonlinear Inequalities

In this chapter, we have 4 types of equations, for which we need to:

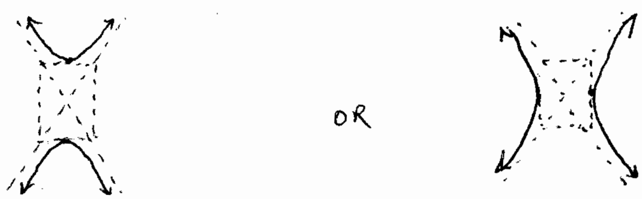
- Notice algebraic features that are important
- Identify what shape goes with which type of equation

Parabola: has one squared variable x^2 and y or OR y^2 and x or

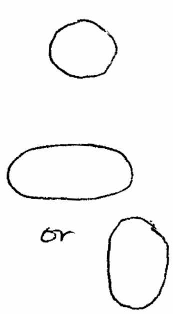


Hyperbola: has both x^2 and y^2, subtracted

one hyperbola has two branches



Circle and Ellipse: both have x^2 and y^2, added, making a round object.



→ circles have the same coefficients for x^2 and y^2

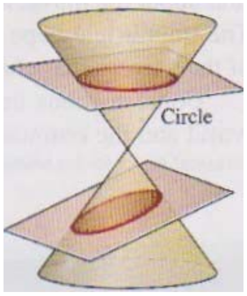
$ax^2 + ay^2$

→ ellipses have different coefficients for x^2 and y^2

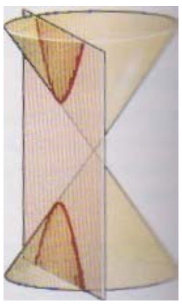
$ax^2 + by^2$

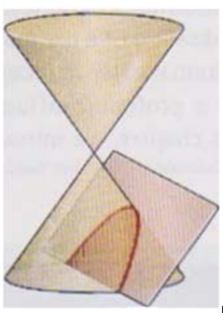
(Though a and b are often written as fractions!)

These four shapes are called conics or conic sections because the occur by slicing a double cone (see picture next page).



Circle above, Ellipse below


(one) Hyperbola (with two branches)


Parabola

Math 70: Parabola and Circle

Objectives:

1) Given two points, calculate the coordinates of the midpoint using the midpoint formula
   $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. [Average the x-coordinates, average the y-coordinates.]
2) Given two points, calculate the distance between them using the distance formula
   $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ [Pythagorean theorem, solve for c.]
3) Graph parabolas
   a. Study the equation's structure first, determine if it's up/down (chap 8) or left/right (chap 10)!
   b. Review up/down parabolas which are functions $y = a(x-h)^2 + k$ or $y = ax^2 + bx + c$
      - Vertex (h,k) or $h = -\frac{b}{a}$ and $k = y(h)$
      - $a > 0$ opens up, $a < 0$ opens down
      - $|a| = 1$ standard shape, $|a| > 1$ narrower than standard, $|a| < 1$ wider than standard
      - Equation of the axis of symmetry $x = h$
   c. Graph parabolas that open left or right and are not functions $x = a(y-k)^2 + h$
      - Swap x and y!!!
      - Vertex (h,k) with h outside the parentheses, and k inside the parentheses (reversed!)
      - Vertex formula $k = -\frac{b}{a}$ (y-coordinate!) and $h = x(h)$
      - $a > 0$ opens right, $a < 0$ opens left
      - $|a| = 1$ standard shape, $|a| > 1$ narrower than standard, $|a| < 1$ wider than standard
      - Equation of the axis of symmetry $y = k$ (horizontal! Use the y-coordinate of vertex.)
4) Graph circles from equations in standard form $(x-h)^2 + (y-k)^2 = r^2$, where the center is (h,k) and the radius is r.
5) Re-write the general form of an equation of a circle $ax^2 + by^2 + cx + dy + e = 0$ in standard form using completing the square, twice!
6) Given the center and radius, write the equation of a circle in standard form.

Examples and Practice

1) Calculate the midpoint between $\left(-4, \frac{19}{3}\right)$ and $\left(3, -\frac{13}{3}\right)$

2) Calculate the distance between $\left(-4, \frac{19}{3}\right)$ and $\left(3, -\frac{13}{3}\right)$

3) Find the vertex, direction, axis of symmetry of $y = -\frac{1}{2}(x-3)^2 - 4$

4) Sketch graph of $x = y^2$

5) Sketch graph of $x = -\frac{1}{2}(y-1)^2 - 4$

6) Use $x = 2y^2 + 4y + 5$.
    a. Find the vertex.
    b. Rewrite the equation in the form $x = a(y-k)^2 + h$

7) Sketch graph of $(x+3)^2 + (y-1)^2 = 25$

8) Sketch graph of $(x-1)^2 + y^2 = 9$

9) Find the center and radius of $x^2 + y^2 + 4x - 8y - 16 = 0$

10) Find the center and radius of $2x^2 + 2y^2 - \frac{1}{2} = 0$

11) Write the equation of a circle (in standard form) having center $(-7,3)$ and radius $\frac{2}{3}$.

12) Use $x = y^2 + 6y + 2$
    a. Find the x-intercept
    b. Find the y-intercept.

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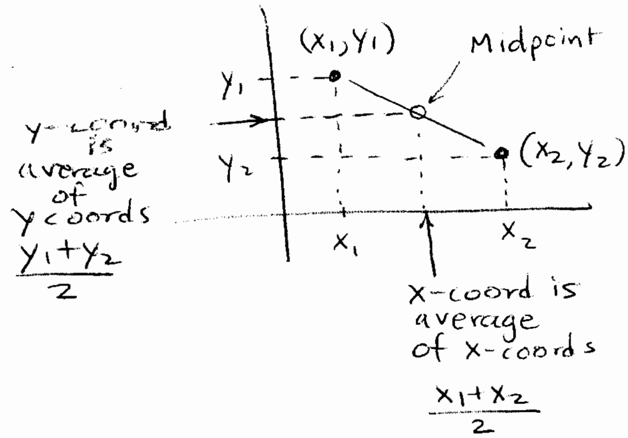
Review:

Midpoint Formula

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

coordinates of a point that's halfway between (x_1,y_1) and (x_2,y_2)



Calculate the midpoint between

SKIP ① $(2,3)$ and $(6,7)$

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$=\left(\frac{2+6}{2}, \frac{3+7}{2}\right)$$

$$=\left(\frac{8}{2}, \frac{10}{2}\right)$$

$$=\boxed{(4,5)}$$

Yes ② $\left(-4, \frac{19}{3}\right)$ and $\left(3, \frac{-13}{3}\right)$

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$=\left(\frac{-4+3}{2}, \frac{19/3+{-13/3}}{2}\right)$$

$$=\left(\frac{-1}{2}, \frac{6/3}{2}\right)$$

$$=\left(-\frac{1}{2}, \frac{2}{2}\right)$$

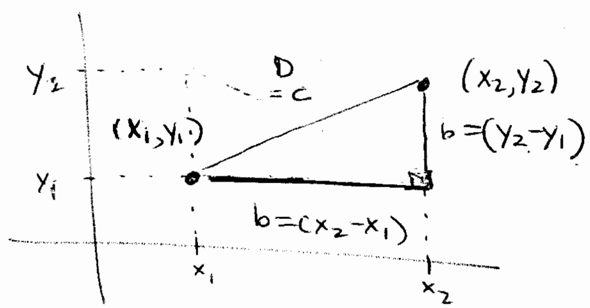
$$=\boxed{\left(-\tfrac{1}{2}, 1\right)}$$

Review:

Distance Formula

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

length of a line segment connecting (x_1,y_1) and (x_2,y_2)



$a^2+b^2=c^2$ Pythagorean Theorem

$\sqrt{a^2+b^2}=c$

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=D$$

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Calculate the distance between the points.

(3) $(2,3)$ and $(5,7)$ SKIP

$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$D=\sqrt{(2-5)^2+(3-7)^2}$

$D=\sqrt{(-3)^2+(-4)^2}$

$D=\sqrt{9+}$

$D=\sqrt{25}$

$\boxed{D=5}$

YES (4) $\left(-4,\frac{19}{3}\right)$ and $\left(3,\frac{-13}{3}\right)$

$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$D=\sqrt{(3--4)^2+\left(\frac{-13}{3}-\frac{19}{3}\right)^2}$

$D=\sqrt{7^2+\left(\frac{32}{3}\right)^2}$

$D=\sqrt{49+\frac{1024}{9}}$

$D=\sqrt{\frac{441}{9}+\frac{1024}{9}}$

$D=\sqrt{\frac{1465}{9}}$

$D=\frac{\sqrt{1465}}{\sqrt{9}}$

$D=\boxed{\frac{\sqrt{1465}}{3}}$

1465

5 293

can't simplify

Remember: Graphing up/down parabolas (chapter 8)

$y=a(x-h)^2+k$

$(h,k)=$ vertex

vertex formula

$h=\frac{-b}{2a}$ $k=$ evaluate at $x=h$

$a>0$ opens up, $a<0$ opens down

(a positive, positive y-direction) (a negative, y-direction negative)

$|a|>1$ narrower than $y=x^2$

$|a|<1$ wider than $y=x^2$

$x=h$ equation of axis of symmetry (vertical line through vertex x coord)

YES (4) Sketch $y=-\frac{1}{2}(x-3)^2-4$

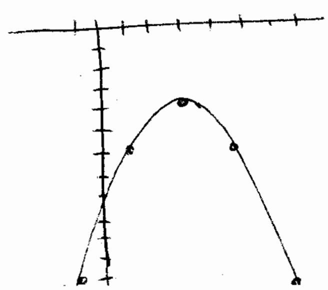
vertex $(3,-4)$

opens down, wide

right 1 down $-\frac{1}{2}(1)^2=-\frac{1}{2}$

right 2 down $-\frac{1}{2}(2)^2=-2$

right 4 down $-\frac{1}{2}(4)^2=-8$



4.5 Sketch $x = -\frac{1}{2}(y-3)^2 - 4$

$x =$ means we have swapped the roles of x and y.

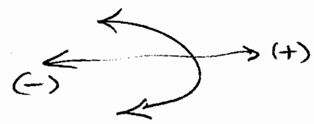
$$x = -\frac{1}{2}(y-3)^2 - 4$$

opposite of y-coordinate of vertex is inside ( ) next to y-variable

x-coordinate of vertex is outside ( ).

vertex $(-4, 3)$.

$a = -\frac{1}{2}$ parabola opens in negative $\underline{\underline{x}}$ direction



parabola opens left.

$|a| = \left|-\frac{1}{2}\right| = \frac{1}{2}$ opens wider than the standard parabola.

x	y
−4	3
	4
	5
	6
	2
	1
	0

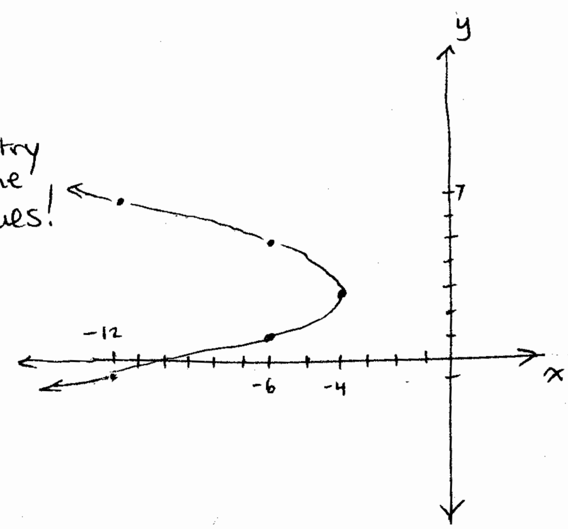
$\left.\begin{matrix}4\\5\\6\end{matrix}\right\}$ up from vertex

$\left.\begin{matrix}2\\1\\0\end{matrix}\right\}$ down from vertex

Complete a table by choosing values of y, starting near the vertex.

x	y
−4	3
−4.5	4
−6	5
−8.5	6
−4.5	2
−6	1
−8.5	0
−12	7
−12	−1

Notice symmetry in the x-values!



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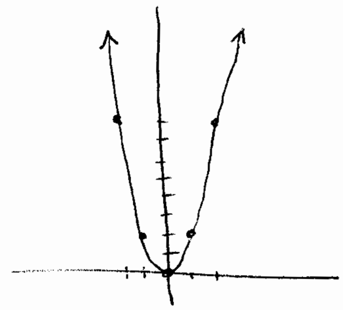
Extra: SKIP (5) Sketch $y = 2x^2$

vertex (0,0)
opens up
narrower

right 1 up $2(1)^2 = 2$
right 2 up $2(2)^2 = 8$
right 3 up $2(3)^2 = 18$



GRAPHING LEFT/RIGHT PARABOLAS

In the equation x and y trade places

$$x = a(y-k)^2 + h$$

CAUTION **

$\boxed{\text{(h,k) is vertex, but h is outside ( ), k is inside next to y.}}$

$a > 0$ opens right — a positive, positive x-direction

$a < 0$ opens left — a negative, negative x-direction

$|a| > 1$ narrower (unchanged)
$|a| < 1$ wider (unchanged)

$y = k$ axis of symmetry is horizontal
(still passes through vertex, but using y-coordinate)

$\boxed{\text{** CAUTION **}}$ vertex formula finds y-coordinate

$$y = k = \frac{-b}{2a} \qquad (x = h \quad \text{evaluate at } y = k)$$

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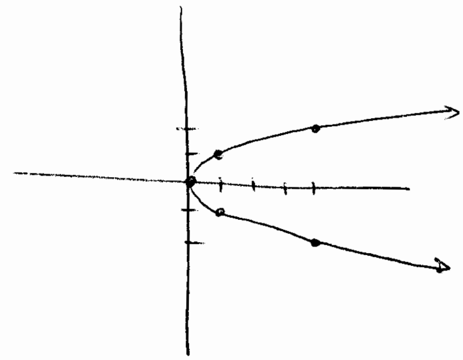
Sketch graphs

YES (6) $x = y^2$

$a = 1$, opens right

vertex $(0,0)$

axis of sym $y = 0$



$\begin{cases} \text{down } 1 & \text{right } 1^2 = 1 \\ \text{up } 1 & \text{right } 1^2 = 1 \end{cases}$

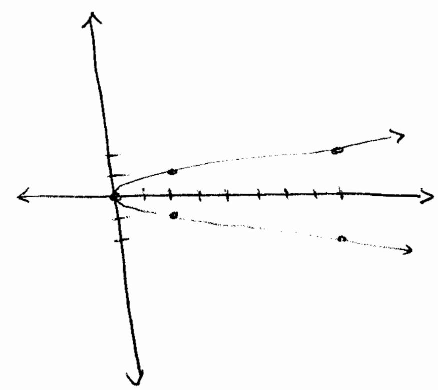
$\begin{cases} \text{down } 2 & \text{right } 2^2 = 4 \\ \text{up } 2 & \text{right } 2^2 = 4 \end{cases}$

SKIP (7) $x = 2y^2$

vertex $(0,0)$

axis of sym $y = 0$

$a = 2$ narrower, opens right



$\begin{cases} \text{down } 1 & \text{right } 2(1)^2 = 2 \\ \text{up } 1 & \text{right } 2(1)^2 = 2 \end{cases}$

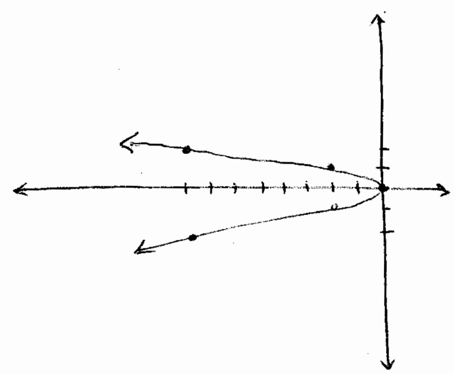
$\begin{cases} \text{down } 2 & \text{right } 2(2)^2 = 8 \\ \text{up } 2 & \text{right } 2(2)^2 = 8 \end{cases}$

SKIP (8) $x = -2y^2$

vertex $(0,0)$

axis of sym $y = 0$

$a = -2$ narrower, opens left



$\begin{cases} \text{down } 1 & \text{left } -2(1)^2 = -2 \\ \text{up } 1 & \text{left } -2(1)^2 = -2 \end{cases}$

$\begin{cases} \text{down } 2 & \text{left } -2(2)^2 = -8 \\ \text{up } 2 & \text{left } -2(2)^2 = -8 \end{cases}$

Once you know the direction and how much to multiply your pattern of squares, it's often easier to just turn your paper (rather than get up/down/left/right confused).

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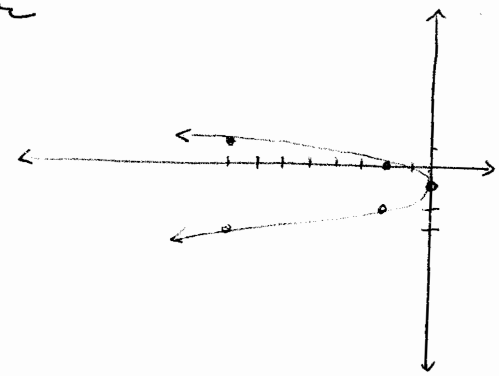
Sketch graphs

9. $x = -2(y+1)^2$ SKIP

vertex $(0, -1)$ y-coord is next to y-variable!

axis of sym $y = -1$

$a = -2$ opens left, narrower

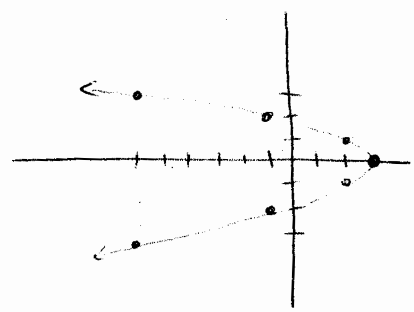


10. $x = -y^2 + 3$ SKIP

vertex $(3, 0)$

axis of sym $y = 0$

$a = -1$ opens left, basic shape

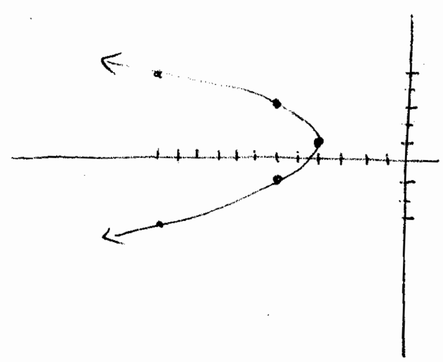


YES 11. $x = -\frac{1}{2}(y-1)^2 - 4$

vertex $(-4, 1)$

axis of sym $y = 1$

$a = -\frac{1}{2}$ opens left, narrower



down 1 left $-\frac{1}{2}(1)^2 = -\frac{1}{2}$ yuck

down 2 left $-\frac{1}{2}(2)^2 = -2$ yes

down 4 left $-\frac{1}{2}(4)^2 = -8$ yes

Study equation's structure first, to determine if its
left/right chapter 10
up/down chapter 8

vertex is most important point, and wrong if you have direction wrong

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Write equations given as $x = ay^2 + by + c$ in the form $x = a(y-k)^2 + h$.

(12) Review: write $y = -x^2 - 2x + 15$ in the form $y = a(x-h)^2 + k$.

SKIP

$y = -(x^2 + 2x \qquad) + 15$ — factor out a from x^2 and x terms

$\# = \frac{2}{2} = 1$

$\#^2 = 1^2 = 1$ — find CTS #, $\#^2$

$y = -(x^2 + 2x + 1) + 15 - (-1)(1)$

add CTS # inside

subtract a · CTS # outside

a CTS#

$\boxed{y = -(x+1)^2 + 16}$ — write perfect square using CTS #

(13) $x = 2y^2 + 4y + 5$

SKIP

$x = 2(y^2 + 2y \qquad) + 5$

$\# = \frac{2}{2} = 1$

$\#^2 = 1^2 = 1$

$x = 2(y^2 + 2y + 1) + 5 - 2(1)$

$\boxed{x = 2(y+1)^2 + 3}$

(14) $x = -y^2 - 4y - 1$

SKIP

$x = -(y^2 + 4y + 4) - 1 - (-1)(4)$

$\# = \frac{4}{2} = 2$

$\#^2 = 2^2 = 4$

$\boxed{x = -(y+2)^2 + 3}$

Math 70 MG 5/e 10.1

Sketch graphs.

(15) $x = 2y^2 + 4y + 5$

YES

find vertex by vertex formula or CTS.

$$y = \frac{-b}{2a}$$

$$y = \frac{-4}{2(2)} = -1$$

$$x = 2(-1)^2 + 4(-1) + 5$$

$$= 2 - 4 + 5$$

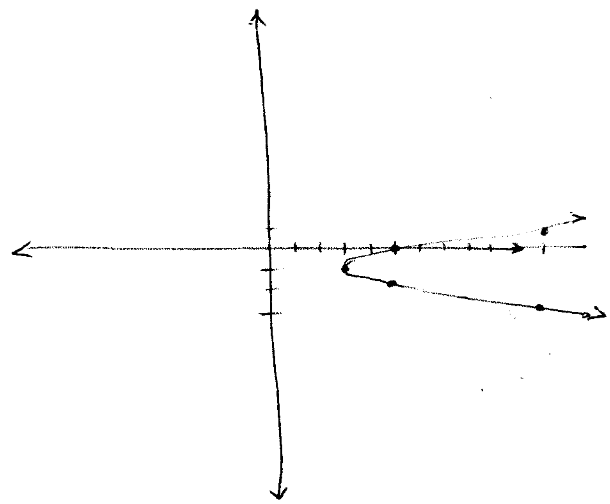
$$= 3$$

vertex $(3, -1)$

[Note: This matches CTS work in (13).]

$a = 2 > 0$ narrower

opens right



Note: You can check left/right graphs in GC by reversing roles of x and y

$y_1 = 2x^2 + 4x + 5$

TABLE

x	y
-1	3
0	5
1	11
-2	5
-3	11

on graph $y = 2x^2 + 4x + 5$

MEANS

x	y
3	-1
5	0
11	1
5	-2
11	-3

on graph $x = 2y^2 + 4y + 5$

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Sketch graphs

SKIP

16 $x = -y^2 - 4y - 1$

find vertex by vertex formula

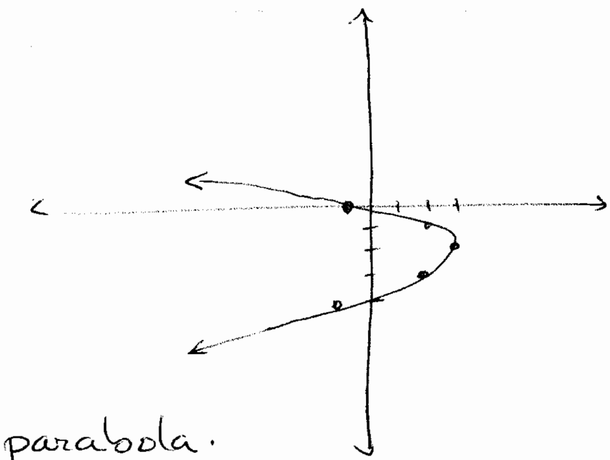
$$y = -\frac{b}{2a} = \frac{-(-4)}{2(-1)} = \frac{4}{-2} = -2$$

$$x = -(-2)^2 - 4(-2) - 1$$
$$= -4 + 8 - 1$$
$$= 3$$

vertex $(3, -2)$

[Note: This matches CTS work in 14.]

$a = -1$ basic shape opens left



SKIP

17 $y = -x^2 - 2x + 15$

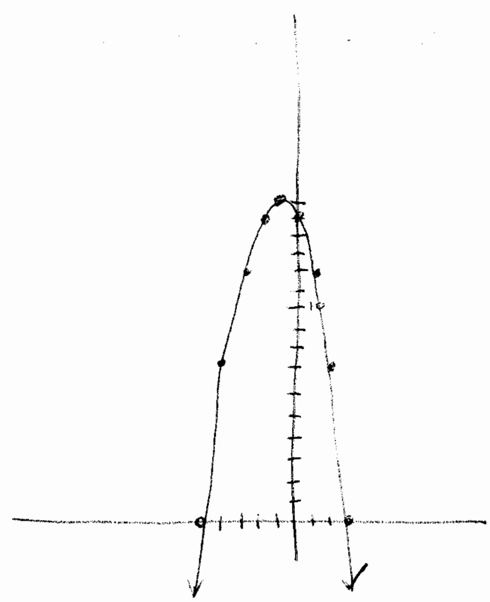
* Notice x's and y's! This is an up/down chap 8 parabola.

vertex $x = -\frac{b}{2a} = \frac{-(-2)}{2(-1)} = \frac{2}{-2} = -1$

$$y = -(-1)^2 - 2(-1) + 15$$
$$= -1 + 2 + 15$$
$$= 16$$

vertex $(-1, 16)$

$a = -1$ opens down basic shape



Rev 8-8-13

Name ______________________________

Date ______________________________

TI-84+ GC 34 Using GC to Graph Parabolae that are Not Functions of x

Objectives: Recall the square root property
Practice solving a quadratic equation for y
Graph the two parts of a horizontal parabola
Identify the axis of symmetry and its equation

When solving for a variable or expression that has been squared, recall that there are usually two solutions. Example: $x^2 = 9$ can be factored to $(x+3)(x-3) = 0$, giving solutions $x = 3, -3$.
The square root property tells us to use $\pm$ when we take the square root of both sides of $x^2 = 9$.

Solve each of the following equations for y by isolating the square and using the square root property.

1) $x = y^2$

2) $x = (y-2)^2$

3) $x = 5(y-2)^2$

4) $x = 5(y-2)^2 - 4$

5) $x = -5(y-2)^2 - 4$

When we graph using the GC, we have only the Y= menu, not an X= menu.
So we must have y as a function of x, not y^2 = an expression in x, or x = an expression in y.
Solve for y first, using algebra. If the square root property was used, write each of the two functions.

6) Graph $x = y^2$ on the GC.

Step 1: Solve the equation for y using the square root property;
$\pm\sqrt{x} = \sqrt{y^2}$ gives $y = \pm\sqrt{x}$.

Step 2: Write as two functions.
$y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$

Step 3: Enter these two functions into the GC Y= menu and graph.

Y= CLEAR 2nd x^2 X,T,θ,n) ENTER

CLEAR (−) 2nd x^2 X,T,θ,n) ENTER ZOOM 6

> The vertex of a horizontal parabola can be found from the standard form of the equation, just as it was for a quadratic function. The roles of x and y are reversed.
> $x = a(y-k)^2 + h$ is the standard form of the equation, and (h,k) is the vertex.
> The axis of symmetry is a horizontal line through the vertex, with equation $y = k$.
> Notice that the y-coefficient of the vertex, k, is inside the parentheses, next to the y variable.
> If a>0, the parabola opens in the positive x-direction, to the right.
> If a<0, the parabola opens in the negative x-direction, to the left.

7) Find the vertex of $x = (y-2)^2$ Does this parabola open left or right?

8) Find the equation of the axis of symmetry.

9) Solve $x = (y-2)^2$ for its two functions. Make a table on the GC for the two functions.

<u>Step 1</u>: Solve the equation for y using the square root property; (Use your previous work.)

<u>Step 2</u>: Write as two functions.

<u>Step 3:</u> Enter these two functions into the GC Y= menu and graph.

[Y=] [CLEAR] [2] [+] [2nd] [x^2] [X,T,θ,n] [)] [ENTER] [CLEAR] [2] [−]
[2nd] [x^2] [X,T,θ,n] [)] [ENTER]

<u>Step 4</u>: Set up the GC table to begin at the vertex.

If the parabola opens left, you'll use the [▲] key to move through the table.

If the parabola opens right, use [▼] to move through the table.

[2nd] [WINDOW] [0] [2nd] [GRAPH]

Use the GC table to fill in this chart:

x-value	$y_1 = 2+\sqrt{x}$	$y_2 = 2-\sqrt{x}$
0		
1		
4		
9		

10) Notice that the x-values provided to you in the previous chart have a pattern. What is it?

11) Find the vertex of $x = 5(y-2)^2$ Does this parabola open left or right?

12) Find the equation of the axis of symmetry of $x = 5(y-2)^2$.

13) Solve $x = 5(y-2)^2$ for its two functions. Make a table on the GC for the two functions.

Step 1: Solve the equation for y using the square root property; (Use your previous work.)

Step 2: Write as two functions.

Step 3: Enter these two functions into the GC Y= menu and graph.

Step 4: Set up the GC table to begin at the vertex.
Use the GC table to fill in this chart:

x-value	$y_1 =$	$y_2 =$
0		
5		
20		
45		

14) Draw axes and label them with an appropriate scale. Plot the points and graph both of the curves, to get the horizontal parabola. Sketch the axis of symmetry.

Graph. You may wish to use these steps:

a. Does this parabola open left or right?
b. What is the vertex?
c. What is the equation of the axis of symmetry?
d. Solve for y, write two functions.
e. Make a table.
f. Draw axes and label them with an appropriate scale. Plot the points and graph parabola. Sketch the axis of symmetry.

15) Graph $x = 5(y-2)^2 - 4$

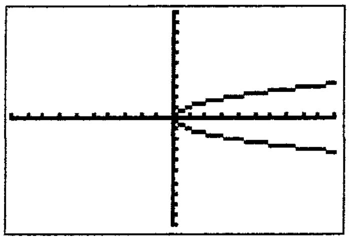
Shortcut: The ordered pairs (x,y) that satisfy $y = -5(x-2)^2 - 4$ can be changed (swap x for y and y for x) to get ordered pairs that satisfy $x = -5(y-2)^2 - 4$. Because $y = -5(x-2)^2 - 4$ is already a function of x, no solving for y is needed. To use this shortcut, skip step d. above, and make a table of values for $y = -5(x-2)^2 - 4$. Then make a second table for graphing $x = -5(y-2)^2 - 4$ by swapping x for y and y for x. If this shortcut doesn't make sense to you, don't use it.

Graph. You may wish to use these steps:

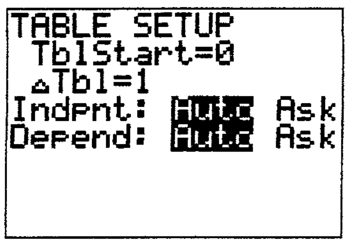
a. Does this parabola open left or right?
b. What is the vertex?
c. What is the equation of the axis of symmetry?
d. Solve for y, write two functions. (Or use the shortcut)
e. Make a table.
f. Draw axes and label them with an appropriate scale. Plot the points and graph parabola. Sketch the axis of symmetry.

16) Graph $x=-5(y-2)^2-4$

1) $y=\pm\sqrt{x}$
2) $y=2\pm\sqrt{x}$
3) $y=2\pm\sqrt{\frac{x}{5}}$ or $y=2\pm\frac{\sqrt{5x}}{5}$
4) $y=2\pm\sqrt{\frac{x+4}{5}}$ or $y=2\pm\frac{\sqrt{5(x+4)}}{5}$
5) $y=2\pm\sqrt{\frac{x+4}{-5}}$ or $y=2\pm\sqrt{\frac{-x-4}{5}}$ or $y=2\pm\frac{\sqrt{-5(x+4)}}{5}$



6)
7) V(0,2) a=1>0, opens right.
8) $y=2$



9) $y=2\pm\sqrt{x}$, $y_1=2+\sqrt{x}$ $y_2=2-\sqrt{x}$ cont

X	Y1	Y2
0	2	2
1	3	1
2	3.4142	.58579
3	3.7321	.26795
4	4	0
5	4.2361	-.2361
6	4.4495	-.4495

X=0

x-value	$y_1=2+\sqrt{x}$	$y_2=2-\sqrt{x}$
0	2	2
1	3	1
4	4	0
9	5	-1

10) The x-values are all perfect squares.

11) V(0,2)

12) $y=2$

13) $y=2\pm\sqrt{\frac{x}{5}}$ or $y=2\pm\frac{\sqrt{5x}}{5}$; $y_1=2+\sqrt{\frac{x}{5}}$, $y_2=2-\sqrt{\frac{x}{5}}$